

# Complete Multi-Representations of Sets in a Computable Measure Space

Yongcheng Wu \*

College of Mathematics and Physics  
Nanjing University of Information Science and Technology  
210044 Nanjing, China  
macswu@163.com

In a recent paper, two multi-representations for the measurable sets in a computable measure space have been introduced, which prove to be topologically complete w.r.t. certain topological properties. In this contribution, we show them recursively complete w.r.t. computability of measure and set-theoretical operations.

## 1 Introduction

In computable analysis, computability concepts depend critically on representations of computational objects. Different representations of a same set of objects can be compared under two kinds of reductions: continuous reductions  $\leq_c$  and computable reductions  $\leq$ , which are string functions transform names under one naming system to names under another one. Most interesting are the complete (multi-)representations among a naturally arising class of naming systems.

**Definition 1.1.** Let  $\Phi$  be a class of naming systems of a set  $X$ . A naming system  $\delta \in \Phi$  is said *topologically/(recursively) complete* in  $\Phi$ , iff  $\phi \leq_c \delta$  resp.  $\phi \leq \delta$  for any  $\phi \in \Phi$ .

For instance, recall that a representation  $\delta$  is said *admissible* w.r.t. a topology  $\tau$  if it is topologically complete among all the continuous representations w.r.t.  $\tau$  and the Cantor topology on strings. Such admissible representations play important role in the topological approach to computable analysis.

Computability frameworks of Lebesgue measure and integration have been addressed by different schools in computable analysis. Ker-I Ko[7] used oracle Turing machines to represent real sets and functions and studied polynomial time complexity of them. Weihrauch[9] investigated computability of measures and integration on the unit interval in the type-2 theory of effectivity. Edalat[3, 4] constructed a domain theoretical framework for Lebesgue measures and integrals. Wu and Weihrauch[17] showed how to construct a measure from an abstract Stone integration. Wu and Ding[15, 16] considered computability of measure and set-theoretical operations in the more general situation of a computable measure space as introduced by [17].

Recently, the author suggests another pair of multi-representations,  $\delta_\mu$  and  $\delta_{\tilde{\mu}}$ , for the measurable sets in a computable measure space. They have been proven to be topologically complete in a certain sense, see [13]. In this paper, we will explore computability of measure and set-theoretical operations w.r.t. them. The results show that  $\delta_\mu$  entails stronger computability than any of the multi-representations applied in [15, 16]. Such results give rise to the recursive completeness of  $\delta_\mu$ . Then we will discuss computability of set operations w.r.t.  $\delta_{\tilde{\mu}}$  and show as a corollary the recursive completeness of  $\delta_{\tilde{\mu}}$ . Proofs of the results are omitted in this extended abstract, which will be given in a separate paper.

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## 2 Preliminaries

### 2.1 Limit relations

A *limit (convergence) relation*, say  $\rightarrow_X$ , on a non-empty set  $X$ , is a relation appointing points in  $X$  to sequences  $(x_n)$  in  $X$ , i.e.  $\rightarrow_X \subseteq X^\omega \times X$ . If  $(x_n) \rightarrow_X x$ , we say that  $(x_n)$  *converges* to  $x$ , where  $(x_n)$  is called a  $\rightarrow_X$ -*convergent sequence* and  $x$  is called a *limit* of  $(x_n)$ .

A pair  $(X, \rightarrow_X)$  will be called a *limit space* if and only if the limit relation  $\rightarrow_X$  on  $X$  satisfies the following three axioms (cf. [6, 8]):

- (L1)  $(x) \rightarrow_X x$ ;
- (L2) If  $(x_n) \rightarrow_X x$  then  $(x_{n_k})_k \rightarrow_X x$ , where  $(x_{n_k})_k$  is a subsequence of  $(x_n)$ ;
- (L3) If  $(x_n)$  is a sequence such that any subsequence of  $(x_n)$  has a subsequence converging to  $x$ , then  $(x_n)$  converges to  $x$ .

Let  $Y$  be a subset of  $X$ . We say that  $Y$  is *dense* in the limit space  $(X, \rightarrow_X)$ , if and only if for every  $x \in X$  there exists a sequence  $(y_n)$  in  $Y$  such that  $(y_n) \rightarrow_X x$ . Limit relations induce a natural notion of continuity: a function  $f : \subseteq X \rightarrow Y$  is said to be *continuous* w.r.t. limit relations  $\rightarrow_X$  and  $\rightarrow_Y$  iff  $f$  preserves convergent sequences (cf. [1, 2]). Sometimes this notion of continuity is called *sequentially continuity* to differ with that defined in terms of topologies.

### 2.2 Computable analysis

We brief here the type-2 theory of effectivity, TTE for short, which is a representation-based approach to computable analysis. Let  $\Sigma$  be a finite alphabet with  $\{0, 1\} \subseteq \Sigma$ . Let  $\Sigma^*$ ,  $\Sigma^\omega$  be the set of finite resp. infinite strings over  $\Sigma$ . On  $\Sigma^*$  we consider the discrete topology  $\tau_*$  and on  $\Sigma^\omega$  the Cantor topology  $\tau_C$  generated by the basis  $\{w\Sigma^\omega \mid w \in \Sigma^*\}$ . In the following content, assume  $W, V, W_i, V_i \in \{\Sigma^*, \Sigma^\omega\}$  for all  $i \in \mathbb{N} = \{0, 1, 2, \dots\}$ . Our computational model is a Turing machine with a one-way output tape. As allowing no revisions on its output it is suitable for computing on infinite strings of symbols. For distinction, we call it a *type-2 machine*, TTM for short. A partial string function  $f : \subseteq W_1 \times W_2 \times \dots \times W_n \rightarrow W_0$  is called *computable* iff it is computed by a TTM. Of course, if the function value is an infinite string, the machine will write step by step each symbol of it and hence never halt. We agree with such an axiom, called the *finiteness property* of TTMs: *Each finite portion of the output is already determined by a finite portion of the input*. This leads to the well-known result found firstly by Grzegorzczuk[5]: Computable string functions are continuous, as formulated by [10, Theorem 2.2.3] in terms of TTE.

Computations on abstract objects are realized by a TTM via naming systems. A naming system for a nonempty set  $X$  is a surjective multi-function  $\delta : \subseteq W \rightrightarrows X$ , which is called a *notation* if  $W = \Sigma^*$  or a *(multi-)representation* if  $W = \Sigma^\omega$ . For any  $w \in \text{dom}(\delta)$ ,  $w$  will be called a  $\delta$ -*name* of  $x \in X$  if and when  $x \in v(w)$ .

**Definition 2.1** (continuity and computability induced by naming systems). Let  $\delta, \gamma$  be naming systems of sets  $X$  and  $Y$  respectively.

1. An element  $x \in X$  is called  $\delta$ -*computable* iff  $x$  has a computable  $\delta$ -name.
2. A subset  $Z \subseteq X$  is called  $\delta$ -*open/r.e./decidable* iff  $\delta^{-1}[Z]$  is open/r.e./decidable.
3. A  $(\delta, \gamma)$ -*realization* of a multi-function  $f : \subseteq X \rightrightarrows Y$  is a (single-valued) string function  $F$  such that  $f(x) \cap \gamma \circ F(u) \neq \emptyset$  for any  $\delta$ -name  $u$  of  $x \in \text{dom}(f)$ .
4. In the above case,  $f$  is called  $(\delta, \gamma)$ -*computable(-continuous)* iff  $F$  is computable(continuous).

5. The above definitions can be extended to Cartesian productions and multi-variable multi-functions in a natural way.

The above definition generalizes the corresponding notions in [10, Definition 3.1.3].

**Definition 2.2** (reducibility between naming systems). Let  $\delta, \gamma$  be two naming systems of a set  $X$ .

1.  $\delta$  is *topologically/recursively reducible* to  $\gamma$ , written  $\delta \leq_t \gamma$  resp.  $\delta \leq \gamma$  iff the identity on  $X$  is  $(\delta, \gamma)$ -continuous/computable. (Equivalent to [11, Definition 24.2])
2.  $\delta <_t \gamma$  denotes that  $\delta \leq_t \gamma$  and  $\gamma \not\leq_t \delta$ . The meaning of  $\delta < \gamma$  is defined accordingly.
3.  $\equiv_t$  and  $\equiv$  denote the equivalences induced by  $\leq_t$  and  $\leq$  respectively.

We take the so-called *tupling function*  $\langle \cdot \rangle$  to encode a finite or infinite sequence of strings as one string (cf. [10, Definition 2.1.7]). For  $w \in \Sigma^*$  and  $q \in W$ ,  $w \sqsubseteq q$  denotes that  $w$  is a prefix of  $q$  and  $w \triangleleft q$  means  $\langle w \rangle$  is a substring of  $q$ .

We will work with the following standard notations  $v_{\mathbb{N}}$  of  $\mathbb{N}$ ,  $v_{\mathbb{Q}}$  of  $\mathbb{Q}$  and standard representations  $\rho$  of  $\mathbb{R}$ ,  $\bar{\rho}$  and  $\bar{\rho}_{>}$  of  $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$  as defined in [10]. A  $\rho$ -name encodes essentially a converging sequence of rational intervals to represent the limit real. Concretely,  $\rho \langle w_1, v_2, w_2, v_2, \dots \rangle = x$  iff  $([v_{\mathbb{Q}}(w_i), v_{\mathbb{Q}}(v_i)])$  is a converging sequence of intervals with a unique limit point  $x$ .  $\rho_C$  is the *Cauchy* representation of  $\mathbb{R}$  which uses a fast Cauchy sequence of rational numbers to represent a real. Concretely,  $\rho_C \langle w_1, w_2, \dots \rangle = x$  iff  $|v_{\mathbb{Q}}(w_i) - x| \leq 2^{-i}$  for any  $i \geq 1$ . In this case the rational sequence will be called a  $\rho_C$ -*expansion* of  $x$ . It is known that  $\rho \equiv \rho_C \equiv \bar{\rho}|^{\mathbb{R}}$ , where  $\bar{\rho}|^{\mathbb{R}}$  denotes  $\bar{\rho}$  restricted to the range  $\mathbb{R}$ .

### 2.3 Measure theory

Let  $\Omega$  be a non-empty set. A *ring* on  $\Omega$  is a collection of subsets of  $\Omega$  closed under the formation of finite unions and differences. An *algebra* (or *field*) on  $\Omega$  is a ring on  $\Omega$  that contains  $\Omega$ . A  $\sigma$ -*algebra* (or  $\sigma$ -*field*) on  $\Omega$  is an algebra on  $\Omega$  which is closed under countable unions. Each set in a  $\sigma$ -algebra is called a *measurable set*. For any class  $\mathcal{C}$  of sets, the minimal  $\sigma$ -algebra including  $\mathcal{C}$  is called the  $\sigma$ -algebra *generated by*  $\mathcal{C}$ , written  $\sigma(\mathcal{C})$ .

Suppose  $\mathcal{A}$  is a  $\sigma$ -algebra on  $\Omega$ . A *measure* on  $\mathcal{A}$  is an extended real function  $\mu : \mathcal{A} \rightarrow [0, \infty]$  which is countably additive. In this case, the triple  $(\Omega, \mathcal{A}, \mu)$  is called a *measure space*.

$A \triangle B := (A \setminus B) \cup (B \setminus A)$  is the *symmetric difference* of sets  $A$  and  $B$ . We write  $A =^* B$  for  $\mu(A \triangle B) = 0$  and  $A \subseteq^* B$  for  $\mu(A - B) = 0$ . We shall use the following terminologies:

1.  $\mathcal{A}_* := \{A \in \mathcal{A} : \mu(A) < \infty\}$ .
2.  $A_{\infty} := \mathcal{A}_*^c$ .
3.  $\mathcal{A}_{\infty\infty} := \{A \in \mathcal{A} : \mu(A) = \mu(A^c) = \infty\}$ .
4.  $\mathcal{A}_{\infty*} := \mathcal{A}_{\infty} - \mathcal{A}_{\infty\infty}$ .
5.  $A_n \nearrow A$  denotes that  $(A_n)$  is an increasing sequence of sets with  $\lim_n A_n = A$ .
6.  $A_n \searrow A$  denotes that  $(A_n)$  is a decreasing sequence of sets with  $\lim_n A_n = A$ .

### 3 Computable measure space

Let  $\Sigma$  be a finite alphabet.  $\Sigma^*$  resp.  $\Sigma^\omega$  denotes the class of finite resp. infinite strings over  $\Sigma$ . We take the discrete topology  $\tau_*$  on  $\Sigma^*$  and the Cantor topology  $\tau_C$  on  $\Sigma^\omega$  (Definition 2.2.2[10]).

**Definition 3.1.** [17] A *computable measure space* is a quintuple  $\mathcal{M} = (\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$  such that

1.  $(\Omega, \mathcal{A}, \mu)$  is a measure space,
2.  $\mathcal{R}$  is a countable ring such that  $\mathcal{A} = \sigma(\mathcal{R})$ ,
3.  $\alpha : \subseteq \Sigma^* \rightarrow \mathcal{R}$  is a notation of  $\mathcal{R}$  with recursive domain,
4.  $\mu$  is  $(\alpha, \rho)$ -computable,
5.  $(A, B) \mapsto A \cup B$  and  $(A, B) \mapsto A - B$  are computable w.r.t.  $\alpha$ .

Therefore, a computable measure space  $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$  is an abstract measure space  $(\Omega, \mathcal{A}, \mu)$  associated with an information structure  $(\mathcal{R}, \alpha)$ , where  $\mathcal{R}$  is a countable ring generating the  $\sigma$ -algebra  $\mathcal{A}$  and  $\alpha$  is a notation of  $\mathcal{R}$  w.r.t. which the measure  $\mu$  and set-theoretical operations restricted to  $\mathcal{R}$  are computable.

**Lemma 3.2.** *In the computable measure space  $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$  with  $\mu(\Omega) = \infty$ , there exists a computable approximate sequence  $(C_n)$  and a computable partition sequence  $(D_n)$  of  $\Omega$  so that*

1. *the multi-function  $E \mapsto n$  with  $E \subseteq C_n$  is  $(\alpha, v_{\mathbb{N}})$ -computable,*
2.  *$\mu(D_n) \geq 2^n$  and  $D_n = C_n - C_{n-1}$  with  $C_0 := \emptyset$ .*

**Proof.** Firstly, let us construct effectively the sequence  $(C_n)$  from the elements in  $\mathcal{R}$ . Since  $\text{dom}(\alpha)$  is recursive, let  $(w_n)$  be a recursive enumeration of  $\text{dom}(\alpha)$ . Let  $k$  be the minimal number such that  $\mu(\bigcup_{i \leq k} \alpha(w_i)) \geq 2$ . Denote  $C_1 := \bigcup_{i \leq k} \alpha(w_i)$ . Suppose for some  $n$  that  $C_i$  is defined for every  $i \leq n$ . Since  $\Omega = \bigcup_i \alpha(w_i)$  and  $\mu(\Omega) = \infty$ , there exists a minimal number  $m$  such that  $\mu(\bigcup_{i \leq m} \alpha(w_i) - C_n) \geq 2^{n+1}$ . Let  $C_{n+1} := \bigcup_{i \leq m} \alpha(w_i)$ . So the sequence  $(C_n)$  is recursively constructed so that, for every  $n \geq 1$ ,

$$C_n \subseteq C_{n+1}, \mu(C_{n+1} - C_n) \geq 2^{n+1} \text{ and } \Omega = \bigcup_n C_n. \quad (3.1)$$

Therefore,  $C_n \nearrow \Omega$ . Furthermore, since the measure, union and difference are computable w.r.t.  $\alpha$  by Definition 3.1, an  $\alpha$ -name of  $C_n$  can be computed for each  $n \geq 1$  and therefore  $(C_n)$  is  $(v_{\mathbb{N}}, \alpha)$ -computable. Let us show claim (1). Given an  $\alpha$ -name  $w$  of some set  $E$ , a number  $n$  can be found s.t.  $w_n = w$  in the recursive sequence  $(w_n)$  as postulated above. This together with (3.1) implies  $E \subseteq C_n$ . Thus claim (1) holds. As for claim (2), it suffices to set  $C_0 := \emptyset$  and  $D_n := C_n - C_{n-1}$  for all  $n \geq 1$ .  $\square$

**Assumption 3.3.** For the remaining content, let  $(\Omega, \mathcal{A}, \mu, \mathcal{R}, \alpha)$  be a computable measure space with  $\mu(\Omega) = \infty$  and  $(C_n), (D_n)$  denote respectively the computable sequences as specified in Lemma 3.2.

### 4 Completeness of the multi-representation $\delta_\mu$

**Definition 4.1.** [13] The limit relation  $(A_n) \rightarrow_\mu A$  for any sequence  $(A_n)$  and set  $A$  in  $\mathcal{A}$  is defined by that

1.  $\lim_n \mu(A_n - A) = 0$ ,
2.  $\lim_n \mu(A \cap B - A_n) = 0$  for any  $B \in \mathcal{A}_*$ .

**Definition 4.2.** [13] The multi-representation  $\delta_\mu : \subseteq \Sigma^\omega \rightrightarrows \mathcal{A}$  is defined by that  $A \in \delta_\mu \langle w_1, w_2, \dots \rangle$  iff the sequence  $(A_n)$  with  $A_n := \alpha(w_n)$  satisfies the following conditions:

1.  $(A_n) \rightarrow_\mu A$ ,
2.  $\forall n < m, \mu(A_n \triangle A_m) \leq 2^{-n}$  whenever  $A \in \mathcal{A}_*$ ,
3.  $\forall n < m, \mu(A_n - A_m) \leq 2^{-n}, \mu(A_m \cap C_n - A_n) \leq 2^{-n}$  and  $\mu(A_n) \geq 2^{-n}$  whenever  $A \in \mathcal{A}_\infty$ .

In this case, the sequence  $(A_n)$  in  $\mathcal{R}$  is called a  $\delta_\mu$ -expansion of  $A$ .

We see that  $\delta_\mu$  uses two different kinds of converging sequences under  $\rightarrow_\mu$  to represent respectively the finite measurable sets  $\mathcal{A}_*$  and the infinitely measurable sets  $\mathcal{A}_\infty$ .

By Lemma 3.2, the computable sequence  $(C_n)$  is indeed a  $\delta_\mu$ -expansion of  $\Omega$ . Since  $(E)$  is a  $\delta_\mu$ -expansion of any  $E \in \mathcal{R}$ , it follows that  $\alpha \leq \delta_\mu$ .

**Theorem 4.3.** [13] *If  $(A_n)$  is a  $\delta_\mu$ -expansion of  $A$ , then  $A =^* \liminf_n A_n =^* \limsup_n A_n$ , where  $=^*$  denotes equality almost anywhere.*

Let  $\rightarrow_{\tau_c}$  denote the limit relation induced by the Cantor topology  $\tau$  on the infinite strings  $\Sigma^\omega$ .  $\delta_\mu$  has the following topological completeness:

**Theorem 4.4.** [13]  *$\delta_\mu$  is topologically complete in the class of  $(\rightarrow_{\tau_c}, \rightarrow_\mu)$ -continuous multi-representations  $\phi$  of  $\mathcal{A}$  such that  $\mathcal{A}_\infty$  is  $\phi$ -open.*

The following lemma guarantees that  $\delta_\mu$  can differentiate effectively  $\mathcal{A}_*$  from  $\mathcal{A}_\infty$ , but cannot differentiate furthermore  $\mathcal{A}_{\infty*}$  from  $\mathcal{A}_{\infty\infty}$  even in the topological sense.

**Lemma 4.5.**

1.  $\Omega$  is  $\delta_\mu$ -computable, i.e. there exists a computable  $\delta_\mu$ -name of  $\Omega$ .
2.  $\mathcal{A}_*$  and  $\mathcal{A}_\infty$  are decidable w.r.t.  $\delta_\mu$ .
3. Both  $\mathcal{A}_{\infty*}$  and  $\mathcal{A}_{\infty\infty}$  are unopen and thus undecidable w.r.t.  $\delta_\mu$ .

In [12], we have shown that it is impossible for any multi-representation  $\psi$  of  $\mathcal{A}$  to make the measure and set-theoretical operations computable on whole  $\mathcal{A}$ . The following theorem is nearly a reformulation of [12, Theorem 4.4].

**Theorem 4.6.** *Let  $\psi : \subseteq \Sigma^\omega \rightarrow \mathcal{A}$  be a multi-representation such that the measure  $\mu$  is  $(\psi, \bar{\rho}_>)$ -continuous. Then for any multi-representations  $\gamma, \delta$  of  $\mathcal{A}$ , the following statements hold:*

1. *Intersection  $\cap$  restricted to  $\mathcal{A}_{\infty\infty}$  is not  $(\gamma, \delta, \psi)$ -continuous.*
2. *Difference  $-$  restricted to  $\mathcal{A}_{\infty\infty}$  is not  $(\gamma, \delta, \psi)$ -continuous.*
3. *Union  $\cup$  and complement  $( )^c$  cannot be continuous w.r.t.  $\psi$  simultaneously.*

*The above statements hold accordingly while “continuous” replaced by “computable”.*

Our studies on computability as well as incomputability of the set operations are included as one theorem:

**Theorem 4.7.**

1.  $\mathcal{A}_\infty$  is  $\delta_\mu$ -decidable.
2. The measure  $\mu$  is  $(\delta_\mu, \bar{\rho})$ -computable.
3. Union  $\cup$  is computable w.r.t.  $\delta_\mu$ .

4. Intersection is  $\delta_\mu$ -computable on  $\{(A, B) : A \in \mathcal{A}_*, \text{ or } B \in \mathcal{A}_*, \text{ or } A \cap B \in \mathcal{A}_\infty\}$ , but not on its complement  $\{(A, B) : A, B \in \mathcal{A}_{\infty\infty} \text{ and } A \cap B \in \mathcal{A}_*\}$ .
5. Difference is  $\delta_\mu$ -computable on  $\{(A, B) : A \in \mathcal{A}_*, \text{ or } A - B \in \mathcal{A}_\infty\}$ , but not on its complement  $\{(A, B) : A \in \mathcal{A}_\infty, \text{ and } A - B \in \mathcal{A}_*\}$ .
6. Complement is  $\delta_\mu$ -computable on  $\mathcal{A}_* \cup \mathcal{A}_{\infty\infty}$ , but not on  $\mathcal{A}_{\infty*}$ .

The above theorem shows that  $\delta_\mu$  entails computability of set operations beyond the domains not being falsified by the more or less general negative results as stated in Theorem 4.6. As a corollary, we have the following completeness theorem of  $\delta_\mu$ :

**Corollary 4.8.**  $\delta_\mu$  is complete in the class  $\Phi_1(\mathcal{A})$  consisting of all naming systems  $\phi$  of  $\mathcal{A}$  such that

1.  $\alpha \leq \phi$ ,
2.  $\mathcal{A}_\infty$  is  $\phi$ -decidable,
3.  $\mu$  is  $(\phi, \overline{\rho})$ -computable,
4. Intersection  $\cap$  is  $\phi$ -computable on  $\mathcal{A}_* \times \mathcal{A}$ ,
5. Symmetric difference  $\triangle$  is  $\phi$ -computable on  $\mathcal{A}_* \times \mathcal{A}_*$ .

This theorem shows that  $\delta_\mu$ -names encode exactly sufficient and necessary information to entail the desired computability of the measure and set-theoretical operations.

## 5 Completeness of the multi-representation $\delta_{\tilde{\mu}}$

The measure  $\mu$  induces the following probability measure  $\tilde{\mu}$ :

$$\tilde{\mu}(A) := \sum_{n=1}^{\infty} \frac{\mu(A \cap D_n)}{\mu(D_n)} 2^{-n} \quad (\forall A \in \mathcal{A}) \quad (5.2)$$

where  $(D_n)$  is the computable partition sequence of  $\Omega$  as assumed in Assumption 3.3.

**Definition 5.1.** [13] The limit relation  $\rightarrow_{\tilde{\mu}} \subseteq \mathcal{A}^\omega \times \mathcal{A}$  is defined by that, for any sequence  $(A_n)$  and set  $A$  in  $\mathcal{A}$ ,  $(A_n) \rightarrow_{\tilde{\mu}} A$  iff  $\lim_n \tilde{\mu}(A_n \triangle A) = 0$ .

**Definition 5.2.** [13] The multi-representation  $\delta_{\tilde{\mu}} : \subseteq \Sigma^\omega \rightrightarrows \mathcal{A}$  is defined by that  $A \in \delta_{\tilde{\mu}} \langle w_1, w_2, \dots \rangle$  iff, for  $A_n := \alpha(w_n)$ ,  $(A_n) \rightarrow_{\tilde{\mu}} A$  and  $\tilde{\mu}(A_n \triangle A_m) \leq 2^{-n}$  for any  $n < m$ . In this case, the sequence  $(A_n)$  on  $\mathcal{R}$  is called a  $\delta_{\tilde{\mu}}$ -expansion of  $A$ .

$\delta_{\tilde{\mu}}$  is admissible w.r.t.  $\rightarrow_{\tilde{\mu}}$ , namely

**Theorem 5.3.** [13]  $\delta_{\tilde{\mu}}$  is topologically complete among the  $(\rightarrow_{\tau_c}, \rightarrow_{\tilde{\mu}})$ -continuous multi-representations of  $\mathcal{A}$ .

**Theorem 5.4.**  $\delta_\mu < \delta_{\tilde{\mu}}$ , i.e.  $\delta_\mu$  is properly reducible to  $\delta_{\tilde{\mu}}$ .

$\delta_{\tilde{\mu}}$  entails the following computability:

**Theorem 5.5.**

1.  $\tilde{\mu}$  is  $(\delta_{\tilde{\mu}}, \rho)$ -computable.
2.  $\mu$  is  $(\delta_{\tilde{\mu}}, \overline{\rho}_<)$ -computable.
3.  $(A, n) \mapsto \mu(A \cap C_n)$  is  $(\delta_{\tilde{\mu}}, v_{\mathbb{N}}, \rho)$ -computable.

4. Each set-theoretical operation is computable w.r.t.  $\delta_{\tilde{\mu}}$ .

As a corollary, we have

**Corollary 5.6.**  $\delta_{\tilde{\mu}}$  is complete in the class  $\Phi_2(\mathcal{A})$  consisting of all naming systems  $\phi$  of  $\mathcal{A}$  such that

1.  $\alpha \leq \phi$ ,
2.  $\tilde{\mu}$  is  $(\phi, \rho)$ -computable,
3. each set-theoretical operation is computable w.r.t.  $\phi$ .

By this corollary, one can see that the equivalence class of  $\delta_{\tilde{\mu}}$  does not depend on the computable sequence  $(D_n)$  employed in the definition of  $\tilde{\mu}$ .

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